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Fixed point theorems for graph preserving multivalued mappings in complete metric spaces with graph

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Abstract

In this paper, we introduce a new type of graph-preserving multivalued mappings in a complete metric space endowed with a directed graph and prove some fixed point theorems under some contractive conditions related to a Reich type contraction. We also give some examples supporting our main results. The main results obtained in the paper extend and generalize many results in the literature.

Keywords: Fixed point, complete metric space, multivalued mapping

บทคัดย่อ

ในการวิจัยนี้ เราได้แนะนำการส่งหลายค่าที่รักษากราฟในปริภูมิเมตริกบริบูรณ์พร้อมด้วยกราฟที่ระบุทิศทาง และได้พิสูจน์ทฤษฎีบทจุดตรึงของการส่งดังกล่าวภายใต้เงื่อนไขการหดตัวบางอย่าง นอกจากนั้นยังได้ให้ตัวอย่างเพื่อสนับสนุนผลลัพธ์หลักที่ได้ศึกษา ผลลัพธ์ที่ได้จากงานวิจัยนี้ ขยายผลลัพธ์อื่น ๆ จำนวนมากตามเอกสารอ้างอิง

คำสำคัญ: จุดตรึง, ปริภูมิเมตริกบริบูรณ์, การส่งหลายค่า

1. Introduction

Fixed point theory of multivalued mappings plays an important role in science and applied science such as Physics and Economics.

Let (X,d) be a metric space, we let $P(X)$ be a power set of X, $P_b(X)$ the set of all nonempty bounded subsets of X, $P_{cb}(X)$ the set of all nonempty closed bounded subsets of X, $P_{cp}(X)$ the set of all nonempty compact subsets of X. If $T : X \longrightarrow P(X)$ is a multivalued mapping, then we will said that x is a *fixed point* of T if $x \in Tx$. The set $Fix(T) := \{x \in X \mid x \in Tx\}$ is called the *fixed point set* of T and $SFix(T) := \{x \in X \mid \{x\} = Tx\}$ is called the *strict fixed point set* of T.

Let A and B be subsets of X, the *Hausdorff distance* of A and B, denoted by $D(A,B)$, is defined by

$$D(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}$$

and the function $\delta(A,B)$ is defined by

$$\delta(A,B) = \sup_{a\in A, b\in B} d(a,b).$$

For $x \in X$ and $A \subseteq X$, the distance between set A and point x is defined by $d(x,A) = \inf_{a\in A} d(x,a)$.

The first well known fixed point theorem for multivalued mappings was established by Nadler in 1969 [1].

Theorem 1.1. *Let (X,d) be a complete metric space and let T be a mapping from X into $P_{cb}(X)$. Assume that there exists $k \in [0,1)$ such that*

$$D(Tx,Ty) \le kd(x,y),\ \forall x,y \in X.$$

Then there exists $z \in X$ such that $z \in Tz$.

The Nadler's fixed point theorem has been extended and modified in many directions. In 2008, Jachymski [2] introduced the concept of G-contraction in complete metric spaces endowed with graph.

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Definition 1.2 ([2]). Let (X, d) be a metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, *i.e.*, $\{(x, x) : x \in X\} \subseteq E(G)$. We say that a mapping $f : X \longrightarrow X$ is a G-contraction if f preserves edges of G, *i.e.*,

$$x, y \in X,\ (x, y) \in E(G) \ \Rightarrow (f(x), f(y)) \in E(G)$$

and there exists $\alpha \in (0, 1)$ such that

$$x, y \in X,\ (x, y) \in E(G) \ \Rightarrow$$
$$d(f(x), f(y)) \leq \alpha d(x, y).$$

Jachymski [2] showed that under the condition on (X, d, G), G-contraction, $f : X \longrightarrow X$ has fixed point if and only if $X_f := \{x \in X \mid (x, f(x)) \in E(G)\}$ is nonempty. The mapping f preserving edge is called a *graph preserving mapping*. Beg *et al.* [5] introduced the concept of G-contraction for multivalued mapping $T : X \longrightarrow P_{cb}(X)$ and proved some fixed point theorems.

Definition 1.3 ([5]). Let $T : X \longrightarrow P_{cb}(X)$ be multivalued mapping. This mapping is said to be a *G-contraction* if there exists $k \in (0, 1)$ such that

$$D(Tx, Ty) \leq kd(x, y),\ \forall (x, y) \in E(G)$$

and if $u \in Tx$ and $v \in Ty$ satisfy that

$$d(u, v) \leq kd(x, y) + \alpha,\ for\ each\ \alpha > 0,$$

then $(u, v) \in E(G)$.

They showed that if (X, d) is complete metric space and (X, d, G) has preserving property, then a G-contraction mapping $T : X \longrightarrow P_{cp}(X)$ has fixed point if and only if $X_f := \{x \in X \mid (x, y) \in E(G)$ for some $y \in Tx\}$ is nonempty.

Throughout this paper, we use $\mathbb{R}_0^+$ to denote the set of all nonnegative real numbers.

Theorem 1.4 (Chifu *et al.* [5]). *Let (X, d) be a complete metric space and G be a directed graph on X such that (X, d, G) satisfies the following property (P):*

$$\forall \{x_n\} \subseteq X \text{ with } x_n \longrightarrow x,\ \exists \text{ subsequence } \{x_{k_n}\} \text{ of } \{x_n\} \text{ such that } (x_{k_n}, x) \in E(G).$$

Let $T : X \rightarrow P_b(X)$ be a multivalued mapping which has the following properties:

(i) *there exist $a, b, c \in \mathbb{R}_0^+$ with $b \neq 0$ and $a+b+c < 1$ such that $\delta(Tx, Ty) \leq ad(x, y) + b\delta(x, Tx) + c\delta(y, Ty)$, $\forall (x, y) \in E(G)$;*

(ii) *for each $x \in X$, the set $\overline{X}_T(x) := \{y \in Tx : (x, y) \in E(G)$ and $\delta(x, Tx) \leq qd(x, y)$ for some $q \in]1, \frac{1-a-c}{b}[\}$ is nonempty.*

Then we have

(a) $Fix(T) = SFix(T) \neq \emptyset$;

(b) if we additionally suppose that $x^*, y^* \in Fix(T) \Rightarrow (x^*, y^*) \in E(G)$, then $Fix(T) = SFix(T) = \{x^*\}$.

In this paper, we modify some conditions of above theorem of Chifu *et al.* to obtain some fixed point theorems for the new type of multivalued mappings in complete metric space with graph.

2. Preliminaries

Let X, Y be two sets and $T : X \longrightarrow P(Y)$. For $M \subseteq X$, we define $T(M) = \bigcup_{x \in M} T(x)$.

Definition 2.1 (Continuity). Let X, Y be topological spaces and $T : X \longrightarrow P(Y)$ a multivalued mapping.

(1) T is called *upper semi-continuous*, if for every $x \in X$ and every open set V in Y with $Tx \subseteq V$, there exists a neighborhood $U(x)$ of x such that $T(U(x)) \subseteq V$.

(2) T is called *lower semi-continuous*, if for every $x \in X$, $y \in Tx$ and every neighborhood $V(y)$ of y, there exists a neighborhood $U(x)$ of x such that

$$Tu \cap V(y) \neq \emptyset,\ \ u \in U(x).$$

(3) T is called *continuous* if it is both upper semi-continuous and lower semi-continuous.

The following result is well known.

Lemma 2.2. *Let (X, d) be a metric space and $T : X \longrightarrow P_{cp}(X)$ is continuous. If $x_n \longrightarrow x$, then $T(x_n) \longrightarrow T(x)$ under the Hausdorff distance.*

3. Fixed point theorems

We start with giving the graph preserving property for multivalued mapping.

Graph preserving property: Let G be a directed graph on X. A mapping $T : X \longrightarrow P(X)$ is said to have *graph preserving property*, if for each $x, y \in X$ if $(x, y) \in E(G)$, then for each $u \in Tx$, there exists $v \in Ty$ such that $(u, v) \in E(G)$.

We first prove the following fixed point theorem.

Theorem 3.1. *Let (X, d) be a complete metric space and G be a directed graph on X such that (X, d, G) satisfies the following property,*
Property (P):

$$\forall \{x_n\} \subseteq X \text{ with } x_n \longrightarrow x, \exists \text{ subsequence } \{x_{k_n}\} \text{ of } \{x_n\} \text{ such that } (x_{k_n}, x) \in E(G).$$

Let $T : X \to P_b(X)$ be multivalued mapping satisfying the following properties:

(i) *there exist $a, b, c \in \mathbb{R}_0^+$ and $a + b + c < 1$ such that $\delta(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + c\delta(y, Ty)$, $\forall (x, y) \in E(G)$;*

(ii) *T has graph preserving property.*

If $X_T = \{x \mid \exists u \in Tx, (x, u) \in E(G)\}$ is nonempty, then $Fix(T) = SFix(T) \neq \emptyset$.

Proof. Since $X_T \neq \emptyset$, there exists $x_0 \in X$, $\exists x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$. So, we get

$$\delta(x_1, Tx_1) \leq \delta(Tx_0, Tx_1).$$

By (i), we have

$$\begin{aligned}\delta(x_1, Tx_1) &\leq \delta(Tx_0, Tx_1)\\ &\leq ad(x_0, x_1) + bd(x_0, Tx_0) + c\delta(x_1, Tx_1)\\ &\leq ad(x_0, x_1) + bd(x_0, x_1) + c\delta(x_1, Tx_1),\end{aligned}$$

then

$$\delta(x_1, Tx_1) \leq \frac{a + b}{1 - c} d(x_0, x_1). \tag{3.1}$$

Since $(x_0, x_1) \in E(G)$ and $x_1 \in Tx_0$, by (ii), there exists $x_2 \in Tx_1$ such that $(x_1, x_2) \in E(G)$. By (3.1), we have

$$d(x_1, x_2) \leq \delta(x_1, Tx_1) \leq \frac{a + b}{1 - c} d(x_0, x_1). \tag{3.2}$$

Since $(x_1, x_2) \in E(G)$, by (i), we have

$$\begin{aligned}\delta(x_2, Tx_2) &\leq \delta(Tx_1, Tx_2)\\ &\leq ad(x_1, x_2) + bd(x_1, Tx_1) + c\delta(x_2, Tx_2)\\ &\leq ad(x_1, x_2) + bd(x_1, x_2) + c\delta(x_2, Tx_2)\\ &\leq \frac{a + b}{1 - c} d(x_1, x_2).\end{aligned}$$

This together with (3.2) we get

$$\delta(x_2, Tx_2) \leq \left(\frac{a + b}{1 - c}\right)^2 d(x_0, x_1). \tag{3.3}$$

Since $(x_1, x_2) \in E(G)$ and $x_2 \in Tx_1$, by (ii), there exists $x_3 \in Tx_2$ such that $(x_2, x_3) \in E(G)$. By (3.3), we have

$$d(x_2, x_3) \leq \delta(x_2, Tx_2) \leq \left(\frac{a + b}{1 - c}\right)^2 d(x_0, x_1). \tag{3.4}$$

By continuing in this way, we obtain sequence $\{x_n\} \subseteq X$ such that

$$(x_n, x_{n+1}) \in E(G),\ \forall n \in \mathbb{N},$$
$$\delta(x_n, Tx_n) \leq \left(\frac{a + b}{1 - c}\right)^n d(x_0, x_1),\ \forall n \in \mathbb{N},$$
$$d(x_n, x_{n+1}) \leq \left(\frac{a + b}{1 - c}\right)^n d(x_0, x_1),\ \forall n \in \mathbb{N}.$$

We claim that $\{x_n\}$ is a Cauchy sequence. Since $\frac{a+b}{1-c} < 1$, we get that $\sum_{i=1}^{\infty} d(x_i, x_{i+1}) < \infty$. This implies that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete metric space, there exists x such that $x_n \longrightarrow x$. We will show that x is a fixed point of T. By property (P), there exists $\{x_{k_n}\} \subseteq \{x_n\}$ such that $(x_{k_n}, x) \in E(G)$, $\forall n \in \mathbb{N}$. Then

$$\begin{aligned}\delta(x, Tx) &\leq d(x, x_{k_n+1}) + \delta(x_{k_n+1}, Tx)\\ &\leq d(x, x_{k_n+1}) + \delta(Tx_{k_n}, Tx)\\ &\leq d(x, x_{k_n+1}) + ad(x_{k_n}, x)\\ &\quad + bd(x_{k_n}, Tx_{k_n}) + c\delta(x, Tx).\end{aligned}$$

It follows that

$$\begin{aligned}\delta(x, Tx) &\leq \frac{1}{1 - c} d(x, x_{k_n+1}) + \frac{a}{1 - c} d(x_{k_n}, x)\\ &\quad + \frac{b}{1 - c}\left(\frac{a + b}{1 - c}\right)^{k_n} d(x_0, x_1).\end{aligned}$$

By taking $n \longrightarrow \infty$, we get $\delta(x, Tx) = 0$. It's obvious that $\emptyset \neq SFix(T) \subseteq Fix(T)$. We will prove that $Fix(T) = SFix(T)$. We need to prove that $Fix(T) \subseteq SFix(T)$. Let $x \in Fix(T)$. By property (P), $(x, x) \in E(G)$, $\forall x \in X$. By (i), we have

$$\begin{aligned}\delta(Tx, Tx) &\leq ad(x, x) + bd(x, Tx) + c\delta(x, Tx)\\ &= c\delta(x, Tx)\\ &\leq c\delta(Tx, Tx).\end{aligned}$$

So $\delta(Tx, Tx)$ must be zero, that is, $Tx = \{x\}$. Thus, $Fix(T) \subseteq SFix(T)$. Hence, $Fix(T) = SFix(T) \neq \emptyset$.

Corollary 3.2. *Let (X, d) be a complete metric space and (X, d, G) have the following properties:*

$$\text{for any } \{x_n\} \text{ in } X, \text{ if } x_n \longrightarrow x \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for } n \in \mathbb{N}, \text{ then there is a subsequence } \{x_{k_n}\} \text{ with } (x_{k_n}, x) \in E(G) \text{ for } n \in \mathbb{N}.$$

Let $f : X \longrightarrow X$ be a G-contraction, and $\emptyset \neq X_f := \{x \in X : (x, fx) \in E(G)\}$ then $Fix(f) \neq \emptyset$.

Proof. It follows directly by Theorem 3.1, with $a = k, b = 0, c = 0$. □

Theorem 3.3. *In addition to the hypothesis of Theorem 3.1, if $x, y \in Fix(T)$ and $(x, y) \in E(G)$, then $Fix(T) = SFix(T) = \{x\}$.*

Proof. By Theorem 3.1, we have $Fix(T) = SFix(T) \neq \emptyset$. Let $x, y \in SFix(T)$. Then

$$\begin{aligned} d(x, y) &= \delta(Tx, Ty) \\ &\leq ad(x, y) + bd(x, Tx) + c\delta(y, Ty) \\ &\leq ad(x, y). \end{aligned}$$

Since $a < 1$, we obtain $d(x, y) = 0$, so $x = y$. Hence $Fix(T) = SFix(T) = \{x\}$. □

Theorem 3.4. *Let (X, d) be a complete metric space and G be a directed graph on X. Let $T : X \rightarrow P_{cp}(X)$ be a continuous multivalued mapping satisfying the following properties:*

(i) *there exist $a, b, c \in \mathbb{R}_0^+$ and $a + b + c < 1$ such that $\delta(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + c\delta(y, Ty)$, $\forall (x, y) \in E(G)$;*

(ii) *T has graph preserving property.*

If $X_T = \{x \mid \exists u \in Tx, (x, u) \in E(G)\}$ is nonempty, then $Fix(T) \neq \emptyset$.

Proof. By using the same proof as that of Theorem 3.1, we can construct a sequence $\{x_n\} \subseteq X$ such that $x_n \longrightarrow x \in X$. We will show that x is a fixed point of T. We have

$$\begin{aligned} d(x, Tx) &\leq d(x, x_k) + d(x_k, Tx_k) + D(Tx_k, Tx) \\ &\leq d(x, x_k) + d(x_k, x_{k+1}) + D(Tx_k, Tx). \end{aligned}$$

By Lemma 2.2, we get $T(x_k) \longrightarrow Tx$. If follow that $d(x, Tx) = 0$. Hence x is fixed point of T, that is $Fix(T) \neq \emptyset$. □

Example 3.1. Let $X = \{1, 1.5, 2, 3\}$ and $G = (X, E(G))$ with $E(G) = \{(1, 1), (1, 1.5), (1, 2), (1.5, 1), (1.5, 2), (2, 1), (2, 3)\}$. Let $T : X \longrightarrow P(X)$ be defined by $T(1) = \{1\}$, $T(1.5) = \{1\}$, $T(2) = \{1, 1.5\}$, $T(3) = \{1, 1.5, 2\}$. It is easy to show that T is continuous and T has graph preserving property, T satisfies property (i) in Theorem 3.4 with $a = 0.9, b = 0.009, c = 0.09$.

Corollary 3.5. *Let (X, d) be complete metric space and $(X, \leq)$ be partially ordered set and let $f : X \longrightarrow X$ be a continuous and non-decreasing mapping such that there exists $k \in [0, 1)$ with*

$$d(f(x), f(y)) \leq kd(x, y), \ \forall x, y \in X, x \geq y.$$

If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

Proof. Let $G = (X, E(G))$ be graph defined by $(u, v) \in E(G)$ if $v \leq u$. Then f satisfies all assumptions of Theorem 3.4. It follows by Theorem 3.4 that f has a fixed point. □

Theorem 3.6. *In addition to the hypothesis of Theorem 3.4, if graph G has loop at every point, then $Fix(T) = SFix(T) \neq \emptyset$.*

Proof. We will prove that $Fix(T) = SFix(T)$. We need to prove that $Fix(T) \subseteq SFix(T)$. Let $x \in Fix(T)$. Then $(x, x) \in E(G)$, $\forall x \in X$. By (i), we have

$$\begin{aligned} \delta(Tx, Tx) &\leq ad(x, x) + bd(x, Tx) + c\delta(x, Tx) \\ &= c\delta(x, Tx) \\ &\leq c\delta(Tx, Tx). \end{aligned}$$

So $\delta(Tx, Tx)$ must be zero, that is, $Tx = \{x\}$. Therefore $Fix(T) \subseteq SFix(T)$. Hence, $Fix(T) = SFix(T) \neq \emptyset$. □

Theorem 3.7. *In addition to the hypothesis of Theorem 3.6 , if $x, y \in Fix(T)$ has property that $(x, y) \in E(G)$, then $Fix(T) = SFix(T) = \{x\}$*

Proof. By Theorem 3.6, we have $Fix(T) = SFix(T) \neq \emptyset$. Let $x, y \in SFix(T)$. Then

$$\begin{aligned} d(x, y) &= \delta(Tx, Ty) \\ &\leq ad(x, y) + bd(x, Tx) + c\delta(y, Ty) \\ &\leq ad(x, y). \end{aligned}$$

Since $a < 1$, we obtain $d(x, y) = 0$, so $x = y$. Hence $Fix(T) = SFix(T) = \{x\}$. □

Remark 3.8. We note that we use δ function to measure distance between the two sets in Theorem 3.1. This function is suitable to the studied graph-preserving multi-valued mappings. How about the Hausdorff distanc, can we use this instead of the δ function? When T in Theorem 3.1 is single, Theorem 3.1 can be viewed as an extension of several known results.

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