Fixed point theorems for graph preserving multivalued mappings in complete metric spaces with graph

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Abstract

In this paper, we introduce a new type of graph-preserving multivalued mappings in a complete metric space endowed with a directed graph and prove some fixed point theorems under some contractive conditions related to a Reich type contraction. We also give some examples supporting our main results. The main results obtained in the paper extend and generalize many results in the literature.

Keywords: Fixed point, complete metric space, multivalued mapping

1. Introduction

Fixed point theory of multivalued mappings plays an important role in science and applied science such as Physics and Economics.

Let $(X, d)$ be a metric space, we let $P(X)$ be a power set of $X$, $P_b(X)$ the set of all nonempty bounded subsets of $X$, $P_{cb}(X)$ the set of all nonempty closed bounded subsets of $X$, $P_{cp}(X)$ the set of all nonempty compact subsets of $X$. If $T : X \rightarrow P(X)$ is a multivalued mapping, then we will said that $x$ is a fixed point of $T$ if $x \in T x$. The set $Fix(T) := \{x \in X | x \in Tx\}$ is called the fixed point set of $T$ and $SFix(T) := \{x \in X | \{x\} = Tx\}$ is called the strict fixed point set of $T$.

Let $A$ and $B$ be subsets of $X$, the Hausdorff distance of $A$ and $B$, denoted by $D(A, B)$, is defined by

$$D(A, B) = \max\left\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\right\}$$

and the function $\delta(A, B)$ is defined by

$$\delta(A, B) = \sup_{a \in A, b \in B} d(a, b).$$

For $x \in X$ and $A \subseteq X$, the distance between set $A$ and point $x$ is defined by $d(x, A) = \inf_{a \in A} d(x, a)$.

The first well known fixed point theorem for multivalued mappings was established by Nadler in 1969 [1].

Theorem 1.1. Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into $P_{cb}(X)$. Assume that there exists $k \in [0, 1)$ such that

$$D(Tx, Ty) \leq kd(x, y), \ \forall x, y \in X.$$ 

Then there exists $z \in X$ such that $z \in Tz$.

The Nadler’s fixed point theorem has been extended and modified in many directions. In 2008, Jachymski [2] introduced the concept of G-contraction in complete metric spaces endowed with graph.
Definition 1.2 ([2]). Let \((X, d)\) be a metric space and let \(G = (V(G), E(G))\) be a directed graph such that \(V(G) = X\) and \(E(G)\) contains all loops, i.e., \(\{(x, x) : x \in X\} \subseteq E(G)\). We say that a mapping \(f : X \rightarrow X\) is a \(G\)-contraction if \(f\) preserves edges of \(G\), i.e.,

\[x, y \in X, \; (x, y) \in E(G) \Rightarrow (f(x), f(y)) \in E(G)\]

and there exists \(\alpha \in (0, 1)\) such that

\[d(f(x), f(y)) \leq \alpha d(x, y)\]

Jachymski [2] showed that under the condition on \((X, d, G)\), \(G\)-contraction, \(f : X \rightarrow X\) has a fixed point if and only if \(X_f := \{x \in X \mid (x, f(x)) \in E(G)\}\) is nonempty. The mapping \(f\) preserving edge is called a graph preserving mapping. Beg et al. [5] introduced the concept of \(G\)-contraction for multivalued mapping \(T : X \rightarrow P_{\text{cb}}(X)\) and proved some fixed point theorems.

Definition 1.3 ([5]). Let \(T : X \rightarrow P_{\text{cb}}(X)\) be multivalued mapping. This mapping is said to be a \(G\)-contraction if there exists \(k \in (0, 1)\) such that

\[D(Tx, Ty) \leq kd(x, y), \; \forall (x, y) \in E(G)\]

and if \(u \in Tx\) and \(v \in Ty\) satisfy that

\[d(u, v) \leq kd(x, y) + \alpha, \; \text{for each} \; \alpha > 0,\]

then \((u, v) \in E(G)\).

They showed that if \((X, d)\) is complete metric space and \((X, d, G)\) has preserving property, then a \(G\)-contraction mapping \(T : X \rightarrow P_{\text{cb}}(X)\) has fixed point if and only if \(X_f := \{x \in X \mid (x, f(x)) \in E(G)\}\) for some \(y \in Tx\) is nonempty.

Throughout this paper, we use \(\mathbb{R}_+^*\) to denote the set of all nonnegative real numbers.

Theorem 1.4 (Chifu et al. [5]). Let \((X, d)\) be a complete metric space and \(G\) be a directed graph on \(X\) such that \((X, d, G)\) satisfies the following property \((P)\):

\[\forall \{x_n\} \subseteq X \text{ with } x_n \rightarrow x, \; \exists \text{ subsequence } \{x_{k_n}\} \text{ of } \{x_n\} \text{ such that } (x_{k_n}, x) \in E(G).\]

Let \(T : X \rightarrow P_{\text{cb}}(X)\) be a multivalued mapping which has the following properties:

(i) there exist \(a, b, c \in \mathbb{R}_+^*\) with \(b \neq 0\) and \(a + b + c < 1\) such that \(\delta(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty), \forall (x, y) \in E(G)\);

(ii) for each \(x \in X\), the set \(X_f(x) := \{y \in Tx : (x, y) \in E(G)\text{ and } \delta(x, Tx) \leq qd(x, y)\text{ for some } q \in [1, \frac{1}{1-\alpha}]\}\) is nonempty.

Then we have

(a) \(\text{Fix}(T) = SFix(T) \neq \emptyset\);

(b) if we additionally suppose that \(x^*, y^* \in \text{Fix}(T) \Rightarrow (x^*, y^*) \in E(G)\), then \(\text{Fix}(T) = SFix(T) = \{x^*\}\).

In this paper, we modify some conditions of above theorem of Chifu et al. to obtain some fixed point theorems for the new type of multivalued mappings in complete metric space with graph.

2. Preliminaries

Let \(X, Y\) be two sets and \(T : X \rightarrow P(Y)\). For \(M \subseteq X\), we define \(T(M) = \bigcup_{x \in M} T(x)\).

Definition 2.1 (Continuity). Let \(X, Y\) be topological spaces and \(T : X \rightarrow P(Y)\) a multivalued mapping.

(1) \(T\) is called upper semi-continuous, if for every \(x \in X\) and every open set \(V\) in \(Y\) with \(Tx \subseteq V\), there exists a neighborhood \(U(x)\) of \(x\) such that \(T(U(x)) \subseteq V\).

(2) \(T\) is called lower semi-continuous, if for every \(x \in X, y \in Tx\) and every neighborhood \(V(y)\) of \(y\), there exists a neighborhood \(U(x)\) of \(x\) such that

\[Tu \cap V(y) \neq \emptyset, \; u \in U(x).\]

(3) \(T\) is called continuous if it is both upper semi-continuous and lower semi-continuous.

The following result is well known.

Lemma 2.2. Let \((X, d)\) be a metric space and \(T : X \rightarrow P_{\text{cb}}(X)\) is continuous. If \(x_n \rightarrow x\), then \(T(x_n) \rightarrow T(x)\) under the Hausdorff distance.
3. Fixed point theorems

We start with giving the graph preserving property for multivalued mapping.

**Graph preserving property:** Let $G$ be a directed graph on $X$. A mapping $T : X \rightarrow P(X)$ is said to have graph preserving property, if for each $x, y \in X$ if $(x, y) \in E(G)$, then for each $u \in T x$, there exists $v \in Ty$ such that $(u, v) \in E(G)$.

We first prove the following fixed point theorem.

**Theorem 3.1.** Let $(X, d)$ be a complete metric space and $G$ be a directed graph on $X$ such that $(X, d, G)$ satisfies the following property, Property (P):

\[ \forall \{x_n\} \subset X \text{ with } x_n \rightarrow x, \exists \text{ subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ such that } (x_{n_k}, x) \in E(G). \]

Let $T : X \rightarrow P(X)$ be multivalued mapping satisfying the following properties:

(i) there exist $a, b, c \in \mathbb{R}^+_0$ and $a + b + c < 1$ such that $\delta(T x, T y) \leq a \delta(x, y) + bd(x, T x) + c \delta(y, T y)$, $\forall (x, y) \in E(G)$;

(ii) $T$ has graph preserving property.

If $X_T = \{x \mid \exists u \in T x, (x, u) \in E(G)\}$ is nonempty, then $Fix(T) = SFix(T) \neq \emptyset$.

**Proof.** Since $X_T \neq \emptyset$, there exists $x_0 \in X$, $\exists x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$. So, we get

\[ \delta(x_1, Tx_1) \leq \delta(Tx_0, Tx_1). \]

By (i), we have

\[ \delta(x_1, Tx_1) \leq \delta(Tx_0, Tx_1) \leq a \delta(x_0, x_1) + b d(x_0, Tx_0) + c \delta(x_1, Tx_1) \leq a \delta(x_0, x_1) + b d(x_0, x_1) + c \delta(x_1, Tx_1), \]

then

\[ \delta(x_1, Tx_1) \leq \frac{a + b}{1 - c} d(x_0, x_1). \]

(3.1)

Since $(x_0, x_1) \in E(G)$ and $x_1 \in Tx_0$, by (ii), there exists $x_2 \in Tx_1$ such that $(x_1, x_2) \in E(G)$. By (3.1), we have

\[ d(x_1, x_2) \leq \delta(x_1, Tx_1) \leq \frac{a + b}{1 - c} d(x_0, x_1). \]

(3.2)

This together with (3.2) we get

\[ \delta(x_2, Tx_2) \leq \left(\frac{a + b}{1 - c}\right)^2 d(x_0, x_1). \]

(3.3)

Since $(x_1, x_2) \in E(G)$ and $x_2 \in Tx_1$, by (ii), there exists $x_3 \in Tx_2$ such that $(x_2, x_3) \in E(G)$. By (3.3), we have

\[ d(x_2, x_3) \leq \delta(x_2, Tx_2) \leq \left(\frac{a + b}{1 - c}\right)^2 d(x_0, x_1). \]

(3.4)

By continuing in this way, we obtain sequence \( \{x_n\} \subset X \) such that

\[ (x_n, x_{n+1}) \in E(G), \forall n \in \mathbb{N}, \]

\[ \delta(x_n, Tx_n) \leq \left(\frac{a + b}{1 - c}\right)^n d(x_0, x_1), \forall n \in \mathbb{N}, \]

\[ d(x_n, x_{n+1}) \leq \left(\frac{a + b}{1 - c}\right)^n d(x_0, x_1), \forall n \in \mathbb{N}. \]

We claim that \( \{x_n\} \) is a Cauchy sequence. Since \( \frac{a + b}{1 - c} < 1 \), we get that \( \sum_{i=1}^{\infty} d(x_i, x_{i+1}) < \infty \). This implies that \( \{x_n\} \) is a Cauchy sequence. Since \((X, d)\) is complete metric space, there exists $x$ such that $x_n \rightarrow x$. We will show that $x$ is a fixed point of $T$. By property (P), there exists \( \{x_{n_k}\} \subset \{x_n\} \) such that $(x_{n_k}, x) \in E(G), \forall n \in \mathbb{N}$.

Thus, by (3.2) we get

\[ \delta(x, x_{n_k}) \leq \delta(x, Tx_{n_k}) \leq \left(\frac{a + b}{1 - c}\right)^n d(x_0, x_1). \]

By taking $n \rightarrow \infty$, we get \( \delta(x, Tx) = 0 \). It's obvious that \( \emptyset \neq SFix(T) \subseteq Fix(T) \). We will prove that $Fix(T) = SFix(T)$. We need to prove that $Fix(T) \subseteq SFix(T)$. Let $x \in Fix(T)$. By property (P), $(x, x) \in E(G), \forall x \in X$. By (i), we have

\[ \delta(Tx, Tx) \leq ad(x, x) + bd(x, Tx) + c \delta(x, Tx) \]

\[ = c \delta(x, Tx) \]

\[ \leq c \delta(Tx, Tx). \]

So \( \delta(Tx, Tx) \) must be zero, that is, $Tx = \{x\}$. Thus, $Fix(T) \subseteq SFix(T)$. Hence, $Fix(T) = SFix(T) \neq \emptyset$. 

Vol. 11 No. 2 March - April 2016
Corollary 3.2. Let \((X, d)\) be a complete metric space and \((X, d, G)\) have the following properties:

for any \(\{x_n\}\) in \(X\), if \(x_n \to x\) and \((x_n, x_{n+1}) \in E(G)\) for \(n \in \mathbb{N}\), then there is a subsequence \(\{x_{k_n}\}\) with \((x_{k_n}, x) \in E(G)\) for \(n \in \mathbb{N}\).

Let \(f : X \to X\) be a \(G\)-contraction, and \(\emptyset \neq X_f := \{x \in X : (x, f x) \in E(G)\}\) then \(\text{Fix}(f) \neq \emptyset\).

Proof. It follows directly by Theorem 3.1, with \(a = k, b = 0, c = 0\).

Theorem 3.3. In addition to the hypothesis of Theorem 3.1 if \(x, y \in \text{Fix}(T)\) and \((x, y) \in E(G)\), then \(\text{Fix}(T) = \text{SFix}(T) = \{x\}\).

Proof. By Theorem 3.1, we have \(\text{Fix}(T) = \text{SFix}(T) \neq \emptyset\). Let \(x, y \in \text{SFix}(T)\). Then

\[
d(x, y) = \delta(Tx, Ty) \\
\leq ad(x, y) + bd(x, Tx) + c\delta(y, Ty) \\
\leq ad(x, y).
\]

Since \(a < 1\), we obtain \(d(x, y) = 0\), so \(x = y\). Hence \(\text{Fix}(T) = \text{SFix}(T) = \{x\}\).

Theorem 3.4. Let \((X, d)\) be a complete metric space and \(G\) be a directed graph on \(X\). Let \(T : X \to P_{\text{fin}}(X)\) be a continuous multivalued mapping satisfying the following properties:

(i) there exist \(a, b, c \in \mathbb{R}^+_0\) and \(a + b + c < 1\) such that \(\delta(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + c\delta(y, Ty), \forall (x, y) \in E(G)\);

(ii) \(T\) has graph preserving property.

If \(X_T = \{x \mid \exists u \in Tx, (x, u) \in E(G)\}\) is nonempty, then \(\text{Fix}(T) \neq \emptyset\).

Proof. By using the same proof as that of Theorem 3.1, we can construct a sequence \(\{x_n\} \subseteq X\) such that \(x_n \to x \in X\). We will show that \(x\) is a fixed point of \(T\). We have

\[
d(x, Tx) \leq d(x, x_k) + d(x_k, Tx_k) + D(Tx_k, Tx) \\
\leq d(x, x_k) + d(x_k, x_{k+1}) + D(Tx_k, Tx).
\]

By Lemma 2.4, we get \(T(x_k) \to Tx\). If follow that \(d(x, Tx) = 0\). Hence \(x\) is fixed point of \(T\), that is \(\text{Fix}(T) \neq \emptyset\).

Example 3.1. Let \(X = \{1, 1.5, 2, 3\}\) and \(G = (X, E(G))\) with \(E(G) = \{(1, 1), (1, 1.5), (1, 2), (1.5, 1), (1.5, 2), (2, 1), (2, 3)\}\). Let \(T : X \to P(X)\) be defined by \(T(1) = \{1\}, T(1.5) = \{1\}, T(2) = \{1, 1.5\}, T(3) = \{1, 1.5, 2\}\). It is easy to show that \(T\) is continuous and \(T\) has graph preserving property, \(T\) satisfies property (i) in Theorem 3.4 with \(a = 0.9, b = 0.009, c = 0.09\).

Corollary 3.5. Let \((X, d)\) be complete metric space and \((X, \leq)\) be partially ordered set and let \(f : X \to X\) be a continuous and non-decreasing mapping such that there exists \(k \in [0, 1)\) with

\[
d(f(x), f(y)) \leq kd(x, y), \forall x, y \in X, x \geq y.
\]

If there exists \(x_0 \in X\) with \(x_0 \leq f(x_0)\), then \(f\) has a fixed point.

Proof. Let \(G = (X, E(G))\) be graph defined by \((u, v) \in E(G)\) if \(v \leq u\). Then \(f\) satisfies all assumptions of Theorem 3.4. It follows by Theorem 3.4 that \(f\) has a fixed point.

Theorem 3.6. In addition to the hypothesis of Theorem 3.4, if graph \(G\) has loop at every point, then \(\text{Fix}(T) = \text{SFix}(T) \neq \emptyset\).

Proof. We will prove that \(\text{Fix}(T) = \text{SFix}(T)\). We need to prove that \(\text{Fix}(T) \subseteq \text{SFix}(T)\). Let \(x \in \text{Fix}(T)\). Then \((x, x) \in E(G), \forall x \in X\). By (i), we have

\[
\delta(Tx, Tx) \leq ad(x, x) + bd(x, Tx) + c\delta(x, Tx) \\
= c\delta(x, Tx) \\
\leq c\delta(Tx, Tx).
\]

Since \(\delta(Tx, Tx)\) must be zero, that is, \(Tx = \{x\}\). Therefore \(\text{Fix}(T) \subseteq \text{SFix}(T)\). Hence, \(\text{Fix}(T) = \text{SFix}(T) \neq \emptyset\).

Theorem 3.7. In addition to the hypothesis of Theorem 3.6, if \(x, y \in \text{Fix}(T)\) has property that \((x, y) \in E(G)\), then \(\text{Fix}(T) = \text{SFix}(T) = \{x\}\).

Proof. By Theorem 3.6, we have \(\text{Fix}(T) = \text{SFix}(T) \neq \emptyset\). Let \(x, y \in \text{SFix}(T)\). Then

\[
d(x, y) = \delta(Tx, Ty) \\
\leq ad(x, y) + bd(x, Tx) + c\delta(y, Ty) \\
\leq ad(x, y).
\]

Since \(a < 1\), we obtain \(d(x, y) = 0\), so \(x = y\). Hence \(\text{Fix}(T) = \text{SFix}(T) = \{x\}\).
Remark 3.8. We note that we use $\delta$ function to measure distance between the two sets in Theorem 3.1. This function is suitable to the studied graph-preserving multi-valued mappings. How about the Hausdorff distance; can we use this instead of the $\delta$ function? When $T$ in Theorem 3.1 is single, Theorem 3.1 can be viewed as an extension of several known results.

References


