Analytical solution of magnetic field from a DC source buried in a conductive layered space
ผลเฉลยเชิงวิเคราะห์ของสนามแม่เหล็กจากแหล่งจ่ายไฟฟ้ากระแสตรงซึ่งฝังในปริภูมิแบบระดับชั้นนำไฟฟ้า

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Abstract

An analytical solution of the steady state magnetic field due to a direct current source is derived for the problem of a buried current source and a buried receiver. The model is developed for source and receiver electrodes arbitrarily located in a horizontally stratified layered earth. The generalized Hankel transform is introduced to our problem and analytical result is obtained. Our solution is achieved by solving a boundary value problem in the wavenumber domain and then transforming the solution back to the spatial domain. An inverse problem via the use of the Levenberg-Marquardt optimization technique is introduced for finding the conductivity parameters of the ground. The method leads to very good result and has high speed of convergence.

Keywords: magnetic field, direct current, buried electrode, layered medium, Hankel transform

บทคัดย่อ

ผลเฉลยเชิงวิเคราะห์ของสนามแม่เหล็กในสภาวะคงตัวอันเนื่องจากแหล่งกำเนิดไฟฟ้ากระแสตรงได้ถูกนำเสนอขึ้นสำหรับปัญหาของแหล่งจ่ายไฟฟ้าและตัวรับสัญญาณแบบฝัง แบบจำลองถูกพัฒนาสำหรับขั้วไฟฟ้าซึ่งถูกฝังอยู่ใต้พื้นผิวดินที่มีลักษณะเป็นชั้นขนานกับแนวระดับ การแปลงฮันเกลแบบทั่วไปถูกนำมาใช้ในปัญหาซึ่งทำให้ได้ผลเฉลยเชิงวิเคราะห์ ผลเฉลยดังกล่าวสามารถหาได้โดยการแก้ปัญหาค่าขอบในโดเมนเชิงความถี่แล้วแปลงผลเฉลยกลับมายังโดเมนเชิงปริภูมิ เทคนิคการหาค่าเหมาะที่สุดแบบเลเวนเบิร์ก-มาร์ควอร์ทถูกนำมาใช้ในปัญหาผกผันเพื่อหาพารามิเตอร์ของสภาพนำไฟฟ้าของพื้นโลก วิธีดังกล่าวให้ผลลัพธ์ที่ดีและมีการลู่เข้าของผลเฉลยอย่างรวดเร็ว

คำสำคัญ: สนามแม่เหล็ก, ไฟฟ้ากระแสตรง, ขั้วไฟฟ้าแบบฝัง, ตัวกลางแบบระดับชั้น, การแปลงฮันเกล

Introduction

Magnetometric resistivity (MMR) is a magnetic exploration method that has been used successfully to investigate electrical conductivity structures within the earth. The mathematical and analytical aspects of the MMR method are described in a review paper written by Edwards *et al.* [1]. The characteristic anomalies for an anisotropic earth, for thin and thick dykes, for semicylindrical and hemispherical depressions are described in some detail. Edwards [2] concentrated on estimating the ratio of the magnetic fields below and above a known conductive surface layer to infer the basement resistivity. Inayat-Hussain [3] gave a new proof that the magnetic field outside the horizontally stratified conductor is

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independent of the electrical conductivity. Veitch *et al.* [4] indirectly derived a solution of the magnetic field in a layered earth containing buried electrodes by using Stoke's theorem and Ampère's law, which was presented by Daniels [5]. Unfortunately, these researches are not sufficiently general concerning the magnetic field to be used for many applications.

In this paper, an analytical solution of the magnetic field is derived directly from solving a boundary value problem in the wavenumber domain, similar to the approach used by Edwards [2]. The generalized Hankel transform [6] is introduced to our problem and analytical result is obtained. The inversion process, using the Levenberg-Marquardt algorithm [7], is conducted to estimate the conductivity parameters of the ground.

Model and basic equations

As shown in Figure 1, a geometric model of the earth's structure consists of two conductive half-spaces. The half-space above the ground surface $\left(z<0\right)$ is a region of air, denoted by layer 0. The half-space below the ground surface $\left(z>0\right)$ is an n - layered horizontally stratified layered earth with depths to the layers $h_1, h_2, \ldots, h_{n-1}$ measured from the ground surface (the lowermost layer extending to infinity). An electrode of exciting current I is placed deliberately at the interface $z=h_s$ of layer s and layer $s+1$ $\left(0 \le s \le n-1\right)$ to simplify the mathematics. Each layer has electrical conductivity as a function of depth, i.e., $\sigma_k\left(z\right)$ for layer $0 \le k \le n$.

1. Magnetic field from a DC source in a 1D structure

The general steady state Maxwell's equations in the frequency wave number domain [2] can be used to determine the magnetic field for our problem, namely

$$\nabla \times \mathbf{E} = \mathbf{0} \quad \text{and} \quad \nabla \times \mathbf{H} = \sigma \mathbf{E}\,, \tag{1}$$

where $\mathbf{E}$ is the vector electric field, $\mathbf{H}$ is the vector magnetic field and σ is the electrical conductivity of the medium.

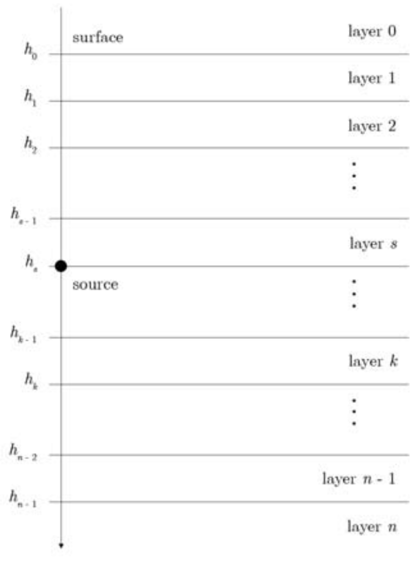

Figure 1 Geometric model of the earth's structure.

Since the problem is axisymmetric and $\mathbf{H}$ has only the azimuthal component in cylindrical coordinates, for simplicity, we use H to represent the azimuthal component in the following derivations. In our study, we denote σ as a function of only depth z, and we now have

$$\frac{\partial^2 H}{\partial z^2}+\sigma\frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right)\frac{\partial H}{\partial z}+\frac{\partial^2 H}{\partial r^2}+\frac{1}{r}\frac{\partial H}{\partial r}-\frac{1}{r^2}H=0\,. \tag{2}$$

The generalized Hankel transform [6] is introduced and defined by

$$\tilde{H}\left(\lambda,z\right)=\int_0^\infty \lambda r H\left(r,z\right)J_1\left(\lambda r\right)dr \quad \text{and} \quad H\left(r,z\right)=\int_0^\infty \tilde{H}\left(\lambda,z\right)J_1\left(\lambda r\right)d\lambda \tag{3}$$

where J_1 is the Bessel function of the first kind of order one and λ is the scaling factor. Taking the transformation on both sides of equation (2), we obtain

$$\frac{\partial^2 \tilde{H}}{\partial z^2}+\sigma\frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right)\frac{\partial \tilde{H}}{\partial z}-\lambda^2\tilde{H}=0\,. \tag{4}$$

Therefore, the magnetic field in each layer can be obtained by taking the inverse generalized Hankel transform to the solution of equation (4).

2. Boundary conditions

The problem satisfies the following physical boundary conditions:

2.1 The azimuthal component of the magnetic field $\tilde{H}_0$ actually converges to zero as z tends to minus infinity.

2.2 The azimuthal component of the magnetic field $\tilde{H}_n$ also converges to zero as z tends to infinity [4].

2.3 The azimuthal component of the magnetic field needs to be continuous on each of the boundary planes except on $z=h_s$, i.e., for each $0\le k\le n-1$ and $k\ne s$,

$$\lim_{z\to h_k^-}\tilde{H}_k=\lim_{z\to h_k^+}\tilde{H}_{k+1}\,. \tag{5}$$

2.4 The radial component of the electric field needs to be continuous on each of the boundary planes [8], i.e., for each $0\le k\le n-1$,

$$\lim_{z\to h_k^-}\frac{1}{\sigma_k}\frac{\partial \tilde{H}_k}{\partial z}=\lim_{z\to h_k^+}\frac{1}{\sigma_{k+1}}\frac{\partial \tilde{H}_{k+1}}{\partial z}\,. \tag{6}$$

2.5 The total current flowing out of any cylindrical surface around a current source must be equal to the current intensity [9], i.e.,

$$2\pi\lim_{h\to 0}\left(\tilde{H}_{s+1}\Big|_{z=h_s+h}-\tilde{H}_s\Big|_{z=h_s-h}\right)=I\,. \tag{7}$$

Solution of the problem

For each layer k, where $0 \le k \le n$ and $n \ge 1$, having constant conductivity (σ is constant), equation (4) reduces to

$$\frac{\partial^2 \tilde{H}_k}{\partial z^2} - \lambda^2 \tilde{H}_k = 0 \tag{8}$$

and the solution is

$$\tilde{H}_k\left(\lambda, z\right) = A_k e^{-\lambda z} + B_k e^{\lambda z}, \tag{9}$$

where the unknown coefficients A_k and B_k are arbitrary constants, which can be determined by using the upward and downward recurrences related to the above boundary conditions (see Sato [9]).

1. 2-layered earth

Consider a 2-layered earth model with a nonconductive layer 0 (representing a region of air). An overburden has a conductivity σ_1 with thickness h_1 overlying a host medium having conductivity σ_2 with infinite depth. A current electrode is located at the interface $z = h_1$. The magnetic field in a host medium can be written as

$$H\left(r, z\right) = \int_0^\infty \frac{I}{2\pi} \frac{\cosh\left(\lambda h_1\right) e^{\lambda\left(h_1 - z\right)}}{\cosh\left(\lambda h_1\right) + \left(\sigma_1/\sigma_2\right)\sinh\left(\lambda h_1\right)} J_1\left(\lambda r\right) d\lambda. \tag{10}$$

2. Uniform half-space

In the case of a uniform half-space, the magnetic field as shown in equation (10) can be determined by using the Lipschitz-Hankel integral [10] and we obtain

$$H\left(r, z\right) = \frac{I}{4\pi r}\left(2 - \frac{z + h_1}{\sqrt{r^2 + \left(z + h_1\right)^2}} - \frac{z - h_1}{\sqrt{r^2 + \left(z - h_1\right)^2}}\right), \tag{11}$$

which is the same result obtained by Veitch *et al.* [4], and Nabighian *et al.* [11].

Inversion process

In our inversion example, we simulate the reflection of magnetic radiation data from our forward model of practical interest. Random errors up to 3% are superimposed on the scaled magnetic fields to simulate the set of real data. Chave's algorithm [12] is used for numerically calculating the inverse Hankel transform of the magnetic field solutions. The example model is a 2-layered electrically conductive earth with current electrode buried at depth h_1 in an overburden of thickness $h_2 > h_1$. Using an artificial interface defined by the plane $z = h_1$, the area of overburden can be divided into two layers, denoted by layers 1 and 2 with conductivity $\sigma_1 = \sigma_2$. A host medium is denoted by layer 3 which has a conductivity σ_3 with infinite depth. The values of the model parameters are tabulated in Table 1. The iterative procedure using the Levenberg-Marquardt method [7] is applied to estimate the model parameters of conductivity variation. Some parameters are assumed to be known and therefore are fixed, e.g., conductivity σ_1. The parameter σ_1 is a conductivity of the earth's surface, which can be known from the measurement. We start the iterative process to find the values of the conductivity parameters with initial guess values $h_2 = 11$ m and $\sigma_3 = 0.01$ S/m. The optimal result of our sample test converges very fast to the true value with percentage errors of h_2 and σ_3 less than 0.03% and 1.49%, respectively, after using only 16 iterations. The graphs of the true and estimated conductivity models are plotted and compared as shown in Figure 2. We clearly see that the graph of the estimated model is close to the true model of conductivity profile. The inversion scheme leads to very good result and has high speed of convergence. This shows the robustness of our model and procedure.

Table 1 Model parameters used in our sample test.

Parameters				
σ_1 (S/m)	σ_2 (S/m)	σ_3 (S/m)	h_1 (m)	h_2 (m)
0.05	0.05	0.5	10	17.5

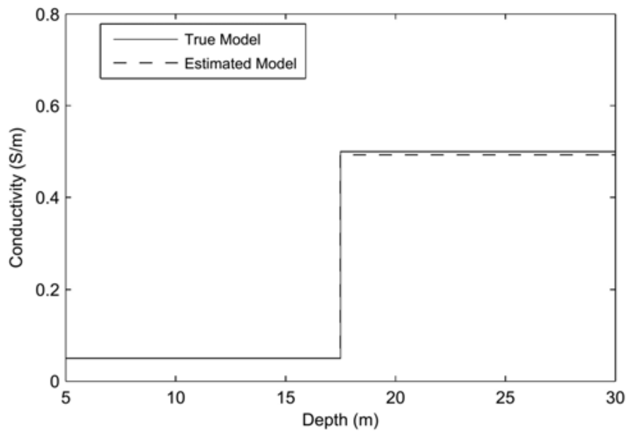

Figure 2 Graphs of conductivity σ against depth z for our inversion example.

Conclusions

We have derived an analytical solution of the steady state magnetic field for the problem of source and receiver electrodes buried anywhere within a horizontally stratified layered earth. Our solution is compared with a published result. This comparison shows that our approach used to handle the physical conditions is correct. The solution can be used to interpret downhole and marine MMR data in which geophysical inversion is required.

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